The Relation Between Offset and Conchoid Constructions

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Abstract

The one-sided offset surface F_d of a given surface F is, roughly speaking, obtained by shifting the tangent planes of F in direction of its oriented normal vector. The conchoid surface G_d of a given surface G is roughly speaking obtained by increasing the distance of G to a fixed reference point O by d. Whereas the offset operation is well known and implemented in most CAD-software systems, the conchoid operation is less known, although already mentioned by the ancient Greeks, and recently studied by some authors.

These two operations are algebraic and create new objects from given input objects. There is a surprisingly simple relation between the offset and the conchoid operation. As derived there exists a rational bijective quadratic map which transforms a given surface F and its offset surfaces F_d to a surface G and its conchoidal surface G_d, and vice versa. Geometric properties of this map are studied and illustrated at hand of some complete examples. Furthermore rational universal parameterizations for offsets and conchoid surfaces are provided.

Keywords: offset surfaces, conchoid surfaces, pedal surface, inverse pedal surface.

1 Introduction

There is a large variety of contributions dealing with the geometry of offsets constructions discussing different aspects; since we are, mainly, focussing on parametrization problems

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we mention here only some of them, see for instance [3], [4], [5], [7], [10], [13], [19] and references on the topic in [18]. Conchoidal constructions, although not so extensively studied, have been recently addressed by different authors too, see for instance [1], [8], [9], [14], [15], [16], [17]. Both geometric constructions were already utilized in the past (Leibnitz studied parallel curves and ancient Greeks used conchoids), and nowadays are used in practical applications (see e.g. [3], [4], [5], [6] and introduction of [14]).

In this paper we show how, using a third geometric construction, namely the foot-point map, one can defined a birational map (i.e. a bijective rational map with rational inverse) between offsets and conchoids. With this we mean that for a given, say surface, F there is another surface G such that the offsets to F are birationally equivalent to the conchoids of G. Therefore, with this birational connection of the two constructions we establish a bridge to translate algebraic-geometric properties, as the genus, the rational parametrizations, etc, from offsets to conchoids and vice-versa.

Throughout this paper, we present the results for surfaces but they are valid for hypersurfaces, in particular for plane algebraic curves.

Let us fix some notation. The scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is denoted as $\mathbf{x} \cdot \mathbf{y}$. Points in $\mathbb{R}^3$ are identified with their coordinate vectors $\mathbf{x} = (x, y, z)$ with respect to a chosen coordinate system. The projective extension of $\mathbb{R}^3$ is denoted as $\mathbb{P}^3$. Points X in $\mathbb{P}^3$ are denoted by their homogeneous coordinate vectors as $X\mathbb{R} = (x_0, x_1, x_2, x_3)\mathbb{R}$ or as $(x_0 : x_1 : x_2 : x_3)$. Let ω be the plane at infinity in $\mathbb{P}^3$, that is the plane of equation $w_0 = 0$. Ideal points, or points at infinity, are represented by $(0, \mathbf{x})\mathbb{R}$, with $\mathbf{x} \in \mathbb{R}^3$. For points $X \notin \omega$, thus $x_0 \neq 0$, the relation between Cartesian and homogeneous coordinates is given by

$$x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}.$$

In the following we briefly recall the offset and conchoidal constructions.

Offset Construction

Consider an irreducible algebraic surface $F \subset \mathbb{R}^3$ given by a (non necessarily rational) parametrization $\mathbf{f}(u, v)$. Let $\mathbf{n}(u, v)$ be the unit normal vector field of $\mathbf{f}(u, v)$, thus $\|\mathbf{n}\| = 1$. Then the one-sided offset surface F_d of F at oriented distance $d \in \mathbb{R}^+$ is given by the parametrization

$$\mathbf{f}_d(u, v) = \mathbf{f}(u, v) + d\mathbf{n}(u, v). \tag{1}$$

Since $\mathbf{n}$ is normalized, its derivatives $\mathbf{n}_u$ and $\mathbf{n}_v$ are orthogonal to $\mathbf{n}$. This implies that the tangent planes of the offset surface F_d are parallel to the tangent planes of F at corresponding points $\mathbf{f}$ and $\mathbf{f}_d$.

Often the offset surface of the surface F is defined as the envelope of a family of spheres of radius d, centered at the base surface F; let us denote this offset notion as $O_d(F)$. This

definition obviously differs from the definition given in (1). Let us analyze the relation between these two concepts. In [13] it is shown that $O_d(F)$ is algebraic and that it has, at most, two irreducible components. If $O_d(F)$ is reducible then the two components are F_d and F_{-d}, the one-sided offsets at distances d and $-d$. If $O_d(F)$ is irreducible, then $O_d(F)$ is the Zariski closure of F_d and also of F_d. In any case, $O_d(F) = F_d \cup F_{-d}$. But, it should be noted that, if $O_d(F)$ is irreducible, the F_d and F_{-d} are not strictly speaking algebraic surfaces, since they represent the external and interior analytic components of the two-sided offset $O_d(F)$. Nevertheless, we will abuse of the terminology, and we will speak about the rationality, etc, meaning the rationality, etc, of their Zariski closure.

Focusing on rational surfaces, we have the following definition.

Definition 1 *A rational surface F is called* rational offset surface *if there exists a rational parametrization $\mathbf{f}(u,v)$ with rational unit normal vector field $\mathbf{n}(u,v)$ of $\mathbf{f}(u,v)$.*

Note that if F is a rational offset surface, then both F_d and F_{-d} admits a rational representation of the type (1). Furthermore,

- if $O_d(F)$ is irreducible then $O_d(F)$ is rational iff F_d admits a rational representation of the type (1) iff F_{-d} a rational representation of the type (1),
- if $O_d(F)$ is reducible then all components of $O_d(F)$ are rational iff F_d and F_{-d} admit a rational representation of the type (1).

Conchoidal Construction

Consider, as above, an irreducible algebraic surface $G \subset \mathbb{R}^3$, $d \in \mathbb{R}^+$, and a fixed reference point O that w.l.o.g. we assume that is the origin of a Cartesian coordinate system. Let G be represented by a polar (non necessarily rational) representation $\mathbf{g}(u,v) = r(u,v)\mathbf{s}(u,v)$, with $\|\mathbf{s}(u,v)\| = 1$. We call $\mathbf{s}(u,v)$ the spherical part of the parameterization $\mathbf{g}(u,v)$ and $r(u,v)$ the radius function. In this situation, the one-sided conchoid surface G_d of G is obtained by increasing the radius $r(u,v)$ by d and thus G_d admits the polar representation

$$\mathbf{g}_d(u,v) = (r(u,v) + d)\mathbf{s}(u,v). \tag{2}$$

In [1], [8], [9], [14], [15], the conchoidal variety is introduced considering both sides. More, precisely, the conchoid surface G_d of G with respect to O at distance d is the (Zariski closure) set, $C_d(G)$, of points Q in the line OP at distance d of a moving point $P \in G$, that is the set

$$C_d(G) = \{Q \in OP \text{ with } P \in G, \text{ and } \overline{QP} = d\}, \tag{3}$$

As in the case of offset surfaces (see above) the two given definitions for conchoids differ, but are clearly related; also for the conchoids it holds that $C_d(G)$ has at most two irreducible components (see [14], [15]). More precisely, if $C_d(G)$ is reducible then the two components are $C_d(G)$ and $C_{-d}(G)$, the one-sided conchoids at distances d and $-d$. If $C_d(G)$ is irreducible, then $C_d(G)$ is the Zariski closure of G_d and also of G_{-d}. In any case, $C_d(F) = G_d \cup G_{-d}$. But, it should be noted that, if $C_d(G)$ is irreducible, the G_d and G_{-d} are not not strictly speaking algebraic surfaces, since they represent the external and interior analytic components of the two-sided conchoid $C_d(F)$. Nevertheless, we will abuse of the terminology, and we will speak about the rationality, etc, meaning the rationality, etc, of their Zariski closure.

To deal with rational surfaces G which have rational conchoid surfaces G_d, we define the following.

Definition 2 *A surface G is called* rational conchoid surface *with respect to the reference point O if G admits a* rational polar representation $r(u,v)\mathbf{s}(u,v)$*, with a rational radius function $r(u,v)$ and a rational parametrization $\mathbf{s}(u,v)$ of the unit sphere S^2.*

Note that if G is a rational conchoid surface, G_d admits the rational representation (2). Furthermore, the same remark on the rationality of the offset, done above, is valid also for conchoids. In this sense, abusing of the terminology, even though F_d and G_d are not, in general, algebraic surfaces, we will speak about their rationality meaning the genus and rationality of their Zariski closure, mainly (or a component of) $O_d(F)$ or $C_d(G)$, respectively.

Contribution: Considering an algebraic irreducible surface F and its continuous family of one-sided offset surfaces F_d, for $d \in \mathbb{R}$, there exists a birational quadratic map α so that the surface $G_d = \alpha(F_d)$ is the one-sided conchoid surface of $G = \alpha(F)$ for $d \in \mathbb{R}$. The inverse map realizes the correspondence between a fixed family of one-sided conchoid surfaces G_d and a family of one-sided offsets surfaces F_d. Since α is a birational map, rationality is preserved in both directions. All geometric properties and results, which are known for one family can be transformed to properties and results for the other family in a simple way. To derive this correspondence, one-sided offset surfaces are considered as envelopes of tangent planes. In addition, we introduce α for implicitly defined algebraic surfaces and, thus, our results extend to the two-sided offsets and conchoids.

2 Representation of Offset and Conchoid surfaces

In this section, we present the representations we will use for the offset and conchoid surfaces. First we will see the initial surface via its dual surface and, then, using parametric representations.

Consider the irreducible algebraic surface $F \subset \mathbb{R}^3$, and let $\overline{F} \subset \mathbb{P}^3$ be its projective closure. We recall that the dual surface $\overline{F}^*$ of $\overline{F}$ is the image of $\overline{F}$ by the gradient map

$$(x_0, \mathbf{x}) \mapsto (u_0, \mathbf{u}) := \nabla(\overline{F})(x_0, \mathbf{x}), \text{ where } \mathbf{x} = (x_1, x_2, x_3), \quad \text{and } \mathbf{u} = (u_1, u_2, u_3).$$

If $\overline{F}$ is given implicitly by the polynomial $\overline{F}(x_0, \mathbf{x})$ then the implicit equation of $\overline{F}^*$ can be computed by eliminating $x_0, \mathbf{x}$ in the algebraic system $\{\overline{F}(x_0, \mathbf{x}) = 0, (u_0, \mathbf{u}) = \nabla(\overline{F})(x_0, \mathbf{x})\}$.

Let us take a closer look at this map. Let us assume that $\overline{F}$ is of class n. That is, we assume that the dual surface $\overline{F}^\star$ has degree n. Note that the class of $\overline{F}$ expresses the algebraically counted number of tangent planes passing through a generic line in $\mathbb{P}^3$, whereas the degree of a surface $\overline{F}$ counts the intersection points with a generic line. Let $\overline{F}^*$ be expressed as

$$\overline{F}^\star(u_0, \mathbf{u}) = u_0^n f_0 + u_0^{n-1} f_1(\mathbf{u}) + \ldots + u_0^{n-j} f_j(\mathbf{u}) + \ldots + u_0 f_{n-1}(\mathbf{u}) + f_n(\mathbf{u}), \tag{4}$$

where $f_j(\mathbf{u})$ are homogeneous polynomials of degree j in $\mathbf{u}$. From this expression, it is clear that the plane at infinity (i.e. $w = 0$) is tangent to $\overline{F}$, or equivalently that $\omega \in \overline{F}^\star$, if $f_0 = 0$. Furthermore, the plane at infinity is an r-fold plane of $\overline{F}$, if $f_0 = \cdots = f_{r-1} = 0$, but $f_r \neq 0$.

For dealing with offset surfaces, we would like to have an affine parametric description of $\overline{F}^*$; we will denote the affine version, with $u_0 = 1$, by F^*. In order to do that, we observe that dual mapping is, essentially, sending points into planes, namely the tangent planes. So, let us assume that we are given a non necessarily rational parametrization $\mathbf{f}(u, v)$ of the affine surface F, and let $\mathbf{n}(u, v)$ be the unit normal vector field of $\mathbf{f}(u, v)$, thus $\|\mathbf{n}\| = 1$. The tangent planes of F are represented by

$$E(u, v) : \mathbf{n}(u, v) \cdot \mathbf{x} = e(u, v), \text{ with } e = \mathbf{f} \cdot \mathbf{n}. \tag{5}$$

Conversely, a two-parameter family of planes $E(u, v)$ in $\mathbb{R}^3$ typically has an envelope surface F. A parametrization $\mathbf{f}$ of F is computed as solution of the system of linear equations

$$\{\mathbf{n} \cdot \mathbf{x} = e, \; \mathbf{n}_u \cdot \mathbf{x} = e_u, \; \mathbf{n}_v \cdot \mathbf{x} = e_v, \}$$

where X_u, X_v denotes the partial derivatives of the function $X(u, v)$ with respect to the variables u and v. Concerning the envelope surface of planes E there exist some degenerate cases. If $\mathbf{n}$ is constant, F is a plane. If $\mathbf{n}$ represents a curve in the unit sphere S^2, the surface F is developable. Otherwise $\mathbf{n}(u, v)$ represents a surface in S^2 and F is typically a non-developable surface in $\mathbb{R}^3$. Even if $\mathbf{n}(u, v)$ is two-dimensional, it might happen that the envelope F of the planes E is a single point, or that the planes E are tangent planes of a curve.

Equation (5) is called a dual (affine parametric) representation of F or an affine parametric representation of F^*. Since $\mathbf{n}$ is normalized, the tangent planes $E_d(u,v)$ of the offset surface F_d, and hence F_d^*, is simply obtained by increasing e by d,

$$E_d(u,v) : \mathbf{n}(u,v) \cdot \mathbf{x} = e(u,v) + d. \tag{6}$$

This is verified by inserting (1) into (6). The function $e(u,v)$ represents the oriented distance of the origin O from the planes $E(u,v)$.

To establish a mapping from a surface F considered as envelope of tangent planes to a surface G, considered as point set, let $\mathcal{P}$ be the point set of $\mathbb{R}^3$ and let $\mathcal{E}$ be the set of planes in $\mathbb{R}^3$.

Parametric representation of offset and conchoid surfaces

Considering the conchoid construction, an irreducible algebraic surface $G \subset \mathbb{R}^3$ is represented by a polar representation $\mathbf{g}(u,v) = r(u,v)\mathbf{s}(u,v)$, with $\|\mathbf{s}(u,v)\| = 1$. Thus G is defined by the map (S^2 is the unit sphere)

$$\begin{array}{rcl} \gamma : S^2 \times \mathbb{R} & \to & \mathcal{P} \\ (\mathbf{s}, r) & \mapsto & \mathbf{g}(u,v) = r(u,v)\mathbf{s}(u,v), \text{ with } \mathbf{s} \in S^2, r \in \mathbb{R}. \end{array} \tag{7}$$

Any rational parametrization $\mathbf{s}(u,v)$ of S^2 and any rational radius function $r(u,v)$ define a rational conchoid surface G in the sense of Definition 2.

Considering the offset construction, an irreducible affine surface F is defined as envelope of the planes $E(u,v) : \mathbf{n}(u,v) \cdot \mathbf{x} = e(u,v)$, with $\|\mathbf{n}\| = 1$ (see above). Thus F is defined by the map

$$\begin{array}{rcl} \varphi : S^2 \times \mathbb{R} & \to & \mathcal{E} \\ (\mathbf{n}, e) & \mapsto & E(u,v) : \mathbf{n}(u,v) \cdot \mathbf{x} = e(u,v), \text{ with } \mathbf{n} \in S^2, e \in \mathbb{R}. \end{array} \tag{8}$$

Any rational parametrization $\mathbf{n}(u,v)$ of S^2 and any rational radius function $e(u,v)$ define a rational offset surface F in the sense of Definition 1. This already indicates that there is a close relation between the offset construction and the conchoid construction. Since the focus is on rational families of offset surfaces and conchoidal surfaces we discuss universal rational parameterizations of S^2.

Following [2] we choose four arbitrary rational functions $a(u,v)$, $b(u,v)$, $c(u,v)$ and $d(u,v)$ without common factor. Let

$$A = 2(ac+bd), B = 2(bc-ad), C = a^2+b^2-c^2-d^2, D = a^2+b^2+c^2+d^2,$$

then $\mathbf{q}(u,v) = \frac{1}{D}(A,B,C)$ is a rational parametrization of the unit sphere S^2. Thus $\varphi(\mathbf{q}(u,v), \rho(u,v))$ with a rational function $\rho(u,v)$ defines a rational parametrization of a rational offset surface and likewise $\gamma(\mathbf{q}(u,v), \rho(u,v))$ defines a rational parametrization of a rational conchoid surface. The only difference is that φ generates a parameterized family of planes $E(u,v)$ whereas γ creates a parameterized family of points $\mathbf{g}(u,v)$ in $\mathbb{R}^3$. How to transform one into the other is discussed in Section 3.

3 The foot-point map

As said in Section 1, we consider the origin O in $\mathbb{R}^3$ as reference point for the conchoidal construction. We now introduce the foot-point map, that will establish the expected connection between offsets and conchoids. The foot-point map α with respect to O is defined as

$$\begin{array}{rcl} \alpha : \mathcal{E} & \to & \mathcal{P} \\ E : \mathbf{x}^T \cdot \mathbf{n} = e & \mapsto & P = \alpha(E) = \dfrac{e}{\|\mathbf{n}\|^2}\mathbf{n}, \end{array} \tag{9}$$

and maps planes $E \subset \mathcal{E}$ to points $P \subset \mathcal{P}$ of $\mathbb{R}^3$. The map is rational and bijective except for planes E passing through O. The inverse map α^{-1} equals the dual map $\alpha^\star$, which reads

$$\begin{array}{rcl} \alpha^\star : \mathcal{P} & \to & \mathcal{E} \\ P = \mathbf{p} & \mapsto & \alpha^\star(P) = E : \mathbf{x}^T \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{p}, \end{array} \tag{10}$$

The maps α and its inverse $\alpha^\star$ are the basic ingredients to construct rational conchoid surfaces from rational offset surfaces and vice versa. In [8], [9], [15] it is shown that the rationality of the conchoid depends on the reference point but the offsets does not. Of course, this dependency is controlled by the foot-point map since it clearly depends on the reference points; indeed if Z is the reference point then α turns to be

$$\begin{array}{rcl} \alpha : \mathcal{E} & \to & \mathcal{P} \\ E : \mathbf{x}^T \cdot \mathbf{n} = e & \mapsto & P = \alpha(E) = Z + \dfrac{e - Z \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}, \end{array}$$

The map (9) is a quadratic plane-to-point mapping. To get more insight into the correspondence between offsets and conchoids we have to study α in more detail. Indeed, we see the map projectively. Planes $U \subset \mathbb{P}^3$ are represented by their homogeneous coordinates $\mathbb{R}(u_0, \ldots, u_3)$, where U is the zero-set in $\mathbb{P}^3$ of the linear equation $U : u_0x_0 + \cdots + u_3x_3 = 0$. Via its coordinates, a plane $U \subset \mathbb{P}^3$ can be identified with a point $\mathbb{R}(u_0, \ldots, u_3)$ in the dual space $\mathbb{P}^{3\star}$. Thus the map $\alpha : \mathcal{E} \to \mathcal{P}$ can be extended to a map $\overline{\alpha} : \mathbb{P}^{3\star} \to \mathbb{P}^3$. Since $\overline{\alpha}$ is uniquely defined by α and since it is not expected that any confusion will appear, both maps will be denoted by α. More precisely, map α is given by

$$\begin{array}{rcl} \alpha : U = \mathbb{R}(u_0, \ldots, u_3) \mapsto \alpha(U) = X & = & (x_0, x_1, x_2, x_3)\mathbb{R} \\ & = & (-(u_1^2 + u_2^2 + u_3^2), u_0u_1, u_0u_2, u_0u_3)\mathbb{R}. \end{array} \tag{11}$$

The baseplanes of α are the ideal plane $\mathbb{R}(1, 0, 0, 0)$, and all planes $\mathbb{R}(u_0, u_1, u_2, u_3)$ satisfying $u_0 = 0, u_1^2 + u_2^2 + u_3^2 = 0$. These are the conjugate complex tangent planes of a quadratic cone with vertex at $O = (1, 0, 0, 0)\mathbb{R}$.

The inverse map $\alpha^{-1} : \mathbb{P}^3 \to \mathbb{P}^{3\star}$ maps points $X \neq O$ to planes $E = \alpha^{-1}(X)$ which have OX as normal and pass through X. Therefore, α is the dual map. Thus, $\alpha^\star : \mathbb{P}^3 \to \mathbb{P}^{3\star}$ is defined as

$$\begin{aligned} \alpha^{-1} = \alpha^\star : X = \mathbb{R}(x_0, \dots, x_3) \mapsto \alpha^\star(X) = U &= (u_0, u_1, u_2, u_3)\mathbb{R}, \\ &= (-(x_1^2 + x_2^2 + x_3^2), x_0x_1, x_0x_2, x_0x_3)\mathbb{R}. \end{aligned} \tag{12}$$

The basepoints of $\alpha^\star$ are the origin $O = (1, 0, 0, 0)\mathbb{R}$ and all points $X = (x_0, x_1, x_2, x_3)\mathbb{R}$ satisfying $j : x_0 = 0, x_1^2 + x_2^2 + x_3^2 = 0$. Note that the conic $j \subset \omega$ does not contain any real point.

The maps α and $\alpha^\star$ can be decomposed in the following way. Consider the inversion $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ at the unit sphere, represented by

$$\sigma : X = \mathbb{R}(x_0, \dots, x_3) \mapsto \sigma(X) = (x_1^2 + x_2^2 + x_3^2, x_0x_1, x_0x_2, x_0x_3)\mathbb{R}. \tag{13}$$

Further, let $\pi : \mathbb{P}^{3\star} \to \mathbb{P}^3$ be the polarity with respect to S^2, with coordinate representation

$$\begin{aligned} \pi : U = \mathbb{R}(u_0, \dots, u_3) &\mapsto \pi(U) = X = (-u_0, u_1, u_2, u_3)\mathbb{R}. \\ \pi^\star : X = (x_0, \dots, x_3)\mathbb{R} &\mapsto \pi^\star(X) = U = \mathbb{R}(-x_0, x_1, x_2, x_3). \end{aligned} \tag{14}$$

It is straight forward that the maps α and $\alpha^\star$ satisfy the following proposition.

Proposition 3 *$\alpha = \sigma \circ \pi$, and $\alpha^\star = \pi \circ \sigma$.*

Since π is a regular map, the base points of α agree with the base points of σ, which are the center $(1, 0, 0, 0)\mathbb{R}$ and the ideal points on the conic $x_0 = 0, x_1^2 + x_2^2 + x_3^2 = 0$.

Given a rational unit vector field $\mathbf{n}(u, v) \subset S^2$ and a rational function $r(u, v)$, the map φ (see (8)) creates a rational offset surface $F^\star$, and the map γ (see (7)) creates a rational conchoid surface G. The relations between α, φ and γ are displayed in the following diagram (15).

$$\begin{tikzcd} & S^2 \times \mathbb{R} \arrow[dl, "\varphi"'] \arrow[dr, "\gamma"] & \\ \mathcal{E} \arrow[rr, "\alpha"'] & & \mathcal{P} \end{tikzcd} \tag{15}$$

Based on the previous reasonings, we have the following theorems.

Theorem 4 *Let $*$ denote the dual map, o^* the offset construction on the dual surface F^* and c the conchoidal construction on surfaces. Then, the following diagram is commutative.*

$$\begin{tikzcd} F \arrow[r, "*"] & F^* \arrow[r, "\alpha"] \arrow[d, "o^*"'] & G \arrow[d, "c"] \\ & F_d^* \arrow[r, "\alpha"'] & G_d \end{tikzcd} \tag{16}$$

From the birationality of the maps in the diagram (16) one deduces the following theorem

Theorem 5 *Let F be an irreducible algebraic surface and $G = \alpha(F^*)$. Then, the offsets F_d of F are birationally equivalent to the conchoids G_d of G.*

The previous theorem implies that all birational invariants are translated from offsets to conchoids and vice-versa. Nevertheless, other properties, as the degree, can not be directly translated. In [12] the degree of offset surfaces is analyzed showing that it increases when offsetting. Although we do not have a theoretical analysis of the degree as in the case of offsets, experiments show that the conchoidal construction also increases the degree. In the next remark we the defining equation of $\alpha(\overline{F}^{\star})$ looks like.

Remark 1 *Let $\overline{F}^*$ be expressed as in (4). By Lemma 3.1. in [11], and by (11) and (12), one deduces that the surface $G = \alpha(\overline{F}^{\star})$ is the zero-set in $\mathbb{P}^3$ of the polynomial, after crossing out the factors (if any) of the the form x_0 and $x_1^2 + x_2^2 + x_3^2$,*

$$\begin{aligned} G(x_0, \mathbf{x}) \;\; = \;\; & (-\mathbf{x}^2)^n f_0 + (-\mathbf{x}^2)^{n-1} x_0 f_1(\mathbf{x}) + \ldots + (-\mathbf{x}^2)^{n-j} x_0^j f_j(\mathbf{x}) + \ldots + \\ & (-\mathbf{x}^2) x_0^{n-1} f_{n-1}(\mathbf{x}) + x_0^n f_n(\mathbf{x}), \end{aligned} \tag{17}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x}$. Say that f_n is expressed as $f_n = \mathbf{x}^\ell h_{n-2\ell}$, with $\ell \geq 0$ and $\gcd(h_{n-2\ell}, \mathbf{x}^2) = 1$. Then if $\ell \neq n$ (that, in particular implies that n is even), the origin $O = (1, 0, 0, 0)\mathbb{R}$ is an $(n - 2\ell)$-fold point of G, whose tangent cone is $h_{n-2\ell}(\mathbf{x}) = 0$. Thus, if $n = 2\ell$ the origin is not on G.

4 A Detailed Example

To illustrate the relations of offset and conchoid construction we present two elementary examples. The first discusses conchoid surfaces of a plane G with respect to a reference point $O \notin G$ and the corresponding offset surfaces, which are offsets of a paraboloid of revolution F with O as focal point. The second example considers offsets of a sphere F and corresponding conchoid surfaces, which are typically rotational surfaces with a Pascal curve as profile, and double point at O.

Conchoid surfaces of a plane

Consider the plane $G : x_3 = 1$ and the reference point $O = (0, 0, 0)$. To compute the conchoid surfaces of G, one might use the trigonometric parameterization $\mathbf{n}(u, v) =$

$(\cos u \cos v, \cos v \sin u, \sin v)$ of S^2. Then G and its conchoid surfaces G_d admit the trigonometric polar representations

$$\begin{aligned} \mathbf{g}(u,v) &= r(u,v)\mathbf{n}(u,v), \text{ and} \\ \mathbf{g}_d(u,v) &= (r(u,v)+d)\mathbf{n}(u,v), \text{ with } r(u,v) = \frac{1}{\sin v}. \end{aligned} \tag{18}$$

The map $\alpha^\star$ maps G to a paraboloid of revolution $F^\star$ and the continuous family of conchoid surfaces G_d is mapped to the family of offset surfaces $F_d^\star$ of the paraboloid $F^\star$. At first, the tangent planes of F and F_d are given by

$$E(u,v) : \mathbf{x} \cdot \mathbf{n}(u,v) = r(u,v), \text{ and } E_d(u,v) : \mathbf{x} \cdot \mathbf{n}(u,v) = r(u,v) + d. \tag{19}$$

At second, the implicit equations of all surfaces being involved will be derived. Using homogeneous point and plane coordinates $X = (x_0, \dots, x_3)$ and $U = (u_0, \dots, u_3)$, respectively, the plane $\overline{G}$ and the paraboloid of revolution $\overline{F}^\star$ are represented by the polynomials

$$\overline{G}(x_0, \mathbf{x}) = (x_3 - x_0) \mapsto F^\star(u_0, \mathbf{u}) = u_0 u_3 + u_1^2 + u_2^2 + u_3^2. \tag{20}$$

The conchoid surfaces $\overline{G}_d$ are are the zero-set of the polynomial

$$\overline{G}_d(x_0, \mathbf{x}) = x_3^2(x_1^2 + x_2^2 + x_3^2) - 2x_3x_0(x_1^2 + x_2^2 + x_3^2) + x_0^2(x_1^2 + x_2^2 + x_3^2(1 - d^2)). \tag{21}$$

Applying $\alpha^\star$ and factorizing out the polynomial $u_0^2(u_1^2+u_2^2+u_3^2)$, the dual implicit equation $\overline{F}_d^\star$ of the offsets of the paraboloid reads

$$\overline{F}_d^\star(u_0, \mathbf{u}) = (u_1^2 + u_2^2)(u_1^2 + u_2^2 - u_3^2 d^2 + 2u_3^2 + 2u_0u_3) + u_3^2((1 - d^2)u_3^2 + 2u_0u_3 + u_0^2). \tag{22}$$

Fig. 1(c) provides an illustration of that example in the 2d-case.

Offset of a sphere

Consider the sphere $F : (x_1 - m)^2 + x_2^2 + x_3^2 - R^2 = 0$. Its offset surfaces F_d are spheres with same center and Radius $R + d$. For a 2d illustration see Fig. 1(b). To establish the correspondence with a family of conchoid surfaces G_d, we have to consider the dual surfaces $F_d^\star$ given by the polynomial

$$\overline{F}_d^\star(u_0, \mathbf{u}) = ((R+d)^2 - m^2)u_1^2 + (R+d)^2(u_2^2 + u_3^2) - 2u_0u_1m - u_0^2.$$

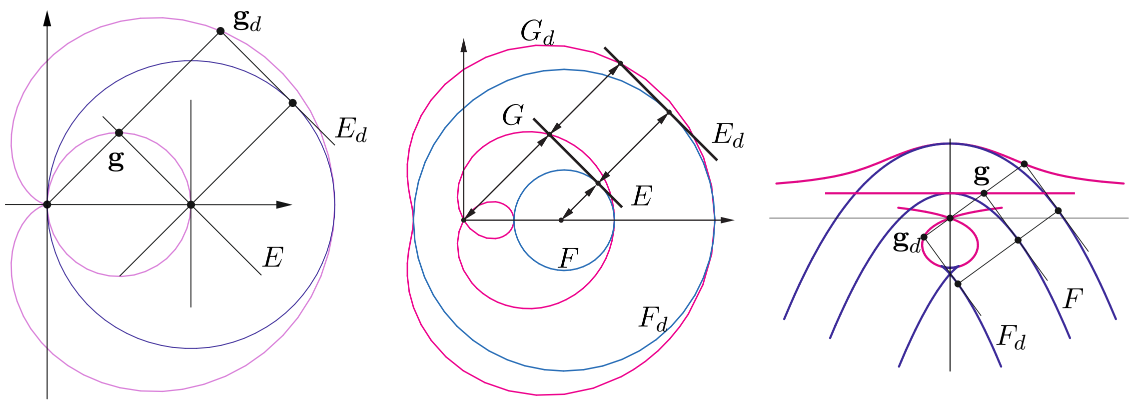
Substituting $d = 0$ gives $\overline{F}^\star$. The bi-rational map α maps surfaces $\overline{F}_d^\star$ to the family of conchoid surfaces $\overline{G}_d$, which are the zero-set of the polynomial

$$\overline{G}_d = x_0^2(x_1^2((R+d)^2 - m^2) + (R+d)^2(x_2^2 + x_3^2)) - (\mathbf{x} \cdot \mathbf{x})^2 + 2mx_0x_1(\mathbf{x} \cdot \mathbf{x}).$$

Substituting $d = 0$ gives $\overline{G}$. Rational offset parameterizations of F_d and rational polar representations of G_d may be derived as follows. Consider again a trigonometric parameterization $\mathbf{n}(u, v) = (\cos u \cos v, \cos v \sin u, \sin v)$ of S^2. Tangent planes E_d of F_d are consequently given by

$$\begin{aligned} E_d(u, v) : \quad & \mathbf{x} \cdot \mathbf{n}(u, v) = r(u, v) + d, \\ \mathbf{g}_d(u, v) \quad = \quad & (r(u, v) + d)\mathbf{n}(u, v), \text{ with } r(u, v) = m \cos u \cos v + R. \end{aligned}$$

We notice that when letting $R = 0$ and $d = 0$, the sphere F degenerates to a bundle of planes passing through the point $M = (m, 0, 0)$. The corresponding 2d-example is illustrated in Fig. 1(a). Its defining polynomial is $F^\star(u_0, \mathbf{u}) = (u_0 + mu_1)$. The corresponding surface via α is a sphere G with OM as diameter, and being the zero-set of $G(x_0, \mathbf{x}) = \mathbf{x} \cdot \mathbf{x} - mx_0x_1$.



(a) Offsets of a pencil and corresponding conchoid curves

(b) Offsets of a circle and corresponding conchoid curves

(c) Offsets of a parabola and conchoids of a line

Figure 1: Relation between offsets and conchoids

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